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Commutator Properties for Periodic Splines

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Commutator properties are established for periodic splines with distinct uniformly spaced knots (on uniform meshes) operated on by certain pseudo-differential operators. The commutation involves the operations of multiplication by a smooth function and application of a discrete version of orthogonal projection obtained by using a quadrature rule (which need integrate only constants exactly) to approximate the inner product. The results mirror a well-known super-approximation property of splines multiplied by smooth functions. © 1999 Academic Press

1. INTRODUCTION

In this paper we establish commutator properties, or local principles, for spaces of periodic splines operated on by pseudo-differential operators. Examples of such relations have been used in the past to study the finite-element method, first by Nitsche and Schatz [8, 9], and in problems such as spline–Galerkin approximations of singular integral equations with nonconstant coefficients in [10, 12, 2, 7]. A general presentation in the context of Toeplitz operators has been given by Hagen, Roch, and Silbermann [6]. Here our commutator property involves multiplication by a smooth function and a discrete version of orthogonal projection. Elsewhere [17] we use the commutator property to study certain spline–qualocation methods in the context of pseudo-differential operator equations with nonconstant

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coefficients (see [16] for the constant coefficient case). The main result is Theorem 2.1.

2. THE MAIN RESULT

First we introduce some notation. We deal with 1-periodic splines on a uniform mesh,

$$x_i := ih, \qquad i = 0, ..., n-1,$$

with $h = 1/n$. A periodic labelling convention allows us to write $x_{i+n} = x_i$ for $i \in \mathbb{Z}$. The space S_h consists of the 1-periodic smoothest splines (i.e., splines with all knots distinct) of order $r \geqslant 1$ on this mesh; that is to say, v_h belongs to S_h if the restriction of v_h to a subinterval (x_i, x_{i+1}) is a polynomial of degree $< r$, and if $v_h \in C^{r-2}(\mathbb{R})$. Similarly, we let S'_h (the "test space") be the space of 1-periodic smoothest splines of order $r' \geqslant 1$.

For each $s \in \mathbb{R}$ the Sobolev norm of a 1-periodic function (or distribution) v may be defined in terms of the Fourier coefficients of v by

$$\|v\|_s := \left(|\hat{v}(0)|^2 + \sum_{k \neq 0} |k|^{2s} \, |\hat{v}(k)|^2 \right)^{1/2} = \left(\sum_{k=-\infty}^{\infty} \max(1, |k|)^{2s} \, |\hat{v}(k)|^2 \right)^{1/2}, \tag{2.1}$$

where

$$\hat{v}(k) := \int_0^1 e^{-2\pi ikx} v(x) \, dx, \qquad k \in \mathbb{Z}.$$

The Sobolev space H^s may be defined as the closure of C^∞ in the norm $\|\cdot\|_s$. In particular, $H^0 = L_2$. (All functions will be assumed 1-periodic unless stated otherwise.)

The L_2 orthogonal projection of $v \in L_2$ onto S'_h is defined by

$$P_h v \in S'_h, \qquad (P_h v, \chi) = (v, \chi) \qquad \forall \chi \in S'_h, \tag{2.2}$$

where, for $v, w \in L_2$,

$$(v, w) := \int_0^1 v(x) \, \overline{w(x)} \, dx. \tag{2.3}$$

We shall be concerned with a discrete variant of the orthogonal projection P_h obtained by replacing the exact integral in the inner product by a composite quadrature rule: Thus we define

$$(v, w)_h := Q_h(v\bar{w}), \tag{2.4}$$

where the quadrature rule Q_h is defined by

$$Q_h(g) := h \sum_{i=0}^{n-1} \sum_{j=1}^{J} \omega_j g(h(\xi_j + i)), \tag{2.5}$$

which is the composite rule obtained by applying a scaled version of the J-point quadrature rule

$$Q(g) := \sum_{j=1}^{J} \omega_j g(\xi_j) \tag{2.6}$$

to each subinterval (x_i, x_{i+1}) of the partition. The quadrature parameters J, $\{\xi_j\}$, $\{\omega_j\}$ are assumed to satisfy

$$J \geqslant 1, \quad 0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_J < 1, \quad \omega_j > 0 \qquad \text{for} \quad j = 1, ..., J,$$
$$\sum_{j=1}^{J} \omega_j = 1,$$

but are otherwise free. Note that $Q_h(g)$ is well defined if $g \in H^s$ with $s > \frac{1}{2}$, since this condition implies that g is continuous.

The quadrature equivalent of P_h, which we shall denote by R_h (and refer to as the discrete orthogonal projection operator), is defined by

$$R_h v \in S'_h, \quad (R_h v, \chi)_h = (v, \chi)_h \qquad \forall \chi \in S'_h. \tag{2.7}$$

Note that P_h and R_h are both projections onto the test space S'_h.

If $r' = 1$ (i.e., the piecewise-constant case) we require that the quadrature points should satisfy also $\xi_1 > 0$, so that there is no quadrature point at the points of discontinuity of the members of S'_h. With this agreement, it is known (see [4, Theorem 3] with $\beta = 0$, L_0 even, $r = r'$) that R_h is well defined by (2.7) for $v \in H^s$, $s > \frac{1}{2}$ if $J \geqslant 2$, and is also well defined if $J = 1$ provided that $\xi_1 \neq \frac{1}{2}$ if r' is even, or $\xi_1 \neq 0$ if r' is odd. In the following we shall assume that the quadrature rule Q satisfies these restrictions.

Now let L denote the pseudo-differential operator of order $\beta \in \mathbb{R}$, with constant coefficients, defined by

$$Lv(x) := (a+b)\, \hat{v}(0) + a \sum_{m \neq 0} |m|^{\beta}\, e^{2\pi i m x} \hat{v}(m)$$
$$+ b \sum_{m \neq 0} (\operatorname{sign} m)\, |m|^{\beta}\, e^{2\pi i m x} \hat{v}(m), \tag{2.8}$$

where a, $b \in \mathbb{R}$.

With these preliminaries, we are now ready to state the main result. Note that in this result fw denotes pointwise multiplication of w by f.

THEOREM 2.1. *Assume* $r > \beta + 1$, *let* δ *satisfy* $0 \leqslant \delta \leqslant 1$, *and assume that* s *and* t *satisfy*

$$\beta - 1 + \delta \leqslant s, \quad t \leqslant r' + s - \delta, \quad t \leqslant r' + \beta - \delta, \quad s < r' + \beta - \tfrac{1}{2}, \quad t < r - \tfrac{1}{2}. \tag{2.9}$$

Then for arbitrary $v > \frac{1}{2}$ *there exists* $C > 0$ *such that for all* $v_h \in S_h$

$$\| R_h(fLv_h) - fR_h Lv_h \|_{s-\beta} \leqslant Ch^{t-s+\delta} \, \|f\|_\kappa \, \|v_h\|_t, \tag{2.10}$$

where

$$\kappa = \max(r' + v, r' + s - \beta + v), \tag{2.11}$$

provided $f \in H^\kappa$.

The theorem is proved in Section 6.

Remark 1. The theorem remains true if the spline trial space S_h is defined to be the trigonometric polynomial space T_h obtained by letting $r \to \infty$. However, it is essential that the test space S'_h remains a spline space.

Remark 2. The theorem remains true with R_h replaced by P_h, and in this case the condition $r > \beta + 1$ can be replaced by $r \geqslant \beta + 1$. This result follows for $s = t = \beta = 0$ from [12] Theorem 2.13 and inequality 2.13(7), by appeal to the identity

$$P_h fLv_h - fP_h Lv_h = [(P_h - I) fP_h + P_h f(I - P_h)] \, Lv_h,$$

together with

$$\|P_h f(I - P_h)\|_{L_2 \to L_2} = \|(I - P_h) fP_h\|_{L_2 \to L_2} \leqslant ch \, \|f\|_{W^{r'}_\infty}. \tag{2.12}$$

The result for other values of s then follows from the inverse inequality and standard approximation arguments. It is not known whether or not (2.10) holds under the weaker condition $r \geqslant \beta + 1$.

3. DISCUSSION

The crucial feature of the results in the theorem is that the power of h appearing on the right-hand side of (2.10) is higher (through the term δ in the exponent) than one would expect from the approximating power of R_h. It is known that the result (2.12) does not hold if S'_h is defined to be a trigonometric polynomial space, rather than a spline space. The same negative result for trigonometric test spaces must also be expected to be true for the results in Theorem 2.1, thus these are very much spline results, which depend on the local character of splines. In the present paper we shall use Fourier series arguments, which hide the local character of splines, instead making the smoothest splines appear almost the same as trigonometric polynomials. It is therefore not surprising that the arguments leading to the commutator properties are technical, rather than transparent. Perhaps a different style of proof is possible, one that captures the local aspect of splines in a more direct way; but at the moment we do not know of such an argument.

It may be thought that the restriction to pseudo-differential operators with constant coefficients as in (2.8) is a serious limitation on the results, but this is not the case, since the results extend also to certain pseudo-differential operators (and to linear combinations of such operators) with nonconstant coefficients, of the form

$$\mathscr{L}v(x) := g(x)\, Lv(x),$$

where g is 1-periodic and suitably smooth. The extension follows from the splitting

$$\begin{aligned}\|R_h(f\mathscr{L}v_h) - fR_h\mathscr{L}v_h\|_{s-\beta} \leqslant &\ \|R_h((fg)\, Lv_h) - (fg)\, R_h Lv_h\|_{s-\beta}\\ &+ \|f(R_h(gLv_h) - gR_h Lv_h)\|_{s-\beta},\end{aligned}$$

which leads to a result for the nonconstant case upon two applications of Theorem 2.1.

4. PROPERTIES OF PERIODIC SMOOTHEST SPLINES

In this section we collect some properties of periodic smoothest splines that will turn out to be useful in the sequel. Most are well known, one (namely Lemma 4.2) is perhaps not. Other properties of periodic splines and spline interpolation are discussed by Golomb [5].

It is known (see [13, 14, 1]) that the smoothest splines of order r, i.e., the elements of S_h, are characterised by the recurrence relation for the Fourier coefficients

$$m^r \hat{v}(m) = \mu^r \hat{v}(\mu) \qquad \text{if} \quad m \equiv \mu, \quad v \in S_h, \tag{4.1}$$

where, here and throughout,

$$m \equiv \mu \Leftrightarrow m - \mu = \ell n \qquad \text{for some} \quad \ell \in \mathbb{Z}. \tag{4.2}$$

It is convenient to use a special basis for S_h, as introduced by Schoenberg [15] and Golomb [5]. Let Λ_h be defined by

$$\Lambda_h := \left\{\mu \in \mathbb{Z}\colon -\frac{n}{2} < \mu \leqslant \frac{n}{2}\right\}.$$

Then the basis $\{\psi_\mu\colon \mu \in \Lambda_h\}$ is defined by

$$\psi_\mu(x) := \begin{cases} 1 & \text{if} \quad \mu = 0, \\ \sum\limits_{m \equiv \mu} \left(\dfrac{\mu}{m}\right)^r e^{2\pi i m x} & \text{if} \quad \mu \in \Lambda_h^*, \end{cases} \tag{4.3}$$

where $\Lambda_h^* := \Lambda_h \backslash \{0\}$. Note that $\psi_\mu \in S_h$, because its Fourier coefficients satisfy the recurrence relation (4.1). If $r = 1$ the Fourier series in (4.3) is not absolutely convergent. Whenever the Fourier series does not converge absolutely the sum is to be understood as the limit of the symmetric partial sums. Of course ψ_μ with $r = 1$ is a piecewise-constant function, with simple discontinuities at $x = kh$, $k \in \mathbb{Z}$. The values at the jumps are to be understood as the means of the left- and right-hand limits.

There is a close relationship between ψ_μ and the trigonometric monomial ϕ_μ, where

$$\phi_m(x) := e^{2\pi i m x}, \qquad m \in \mathbb{Z}.$$

In particular, it follows from definition (4.3) that

$$\psi_\mu(x + h) = e^{2\pi i \mu h} \psi_\mu(x),$$

so that ψ_μ behaves under translation by h exactly like ϕ_μ. (This property corresponds to Eqs. (5.8) and (5.17) in Lecture 3 of [15] for exponential Euler splines.) Also, from (4.3) we see that

$$\hat{\psi}_\mu(\nu) = (\psi_\mu, \phi_\nu) = \delta_{\mu\nu} \qquad \text{for} \quad \mu, \nu \in \Lambda_h,$$

which leads to a simple representation for a spline $v_h \in S_h$ in terms of the basis $\{\psi_\mu\}$,

$$v_h = \sum_{\mu \in \Lambda_h} \hat{v}_h(\mu)\, \psi_\mu, \qquad v_h \in S_h. \tag{4.4}$$

The spline ψ_μ for $\mu \in \Lambda_h^*$ can be rewritten, using (4.3), as

$$\psi_\mu = \sum_{\ell = -\infty}^{\infty} \left(\frac{\mu}{\ell n + \mu}\right)^r \phi_{\ell n + \mu}, \tag{4.5}$$

from which follows

$$\psi_\mu(x) = \phi_\mu(x)\, Z_r\left(nx, \frac{\mu}{n}\right), \qquad x \in \mathbb{R}, \quad \mu \in \Lambda_h, \tag{4.6}$$

where, for $\xi \in \mathbb{R}$,

$$Z_r(\xi, y) := \begin{cases} 1 & \text{if} \quad y = 0, \\ \sum_{\ell=-\infty}^{\infty} \left(\frac{y}{\ell + y}\right)^r e^{2\pi i \ell \xi} & \text{if} \quad 0 < |y| < 1. \end{cases} \tag{4.7}$$

Note that $Z_r(\xi, y)$ is 1-periodic in ξ. Note too that we have chosen to define $Z_r(\xi, y)$ for $y \in (-1, 1)$, instead of restricting y to the "natural" domain $|y| \leqslant \frac{1}{2}$. The larger domain for y corresponds to extending definition (4.3) of ψ_μ to values of μ outside Λ_h, which will turn out to be useful to us later.

The complex-valued function $Z_r(\xi, y)$ may be thought of as describing the extent to which ψ_μ is *not* the trigonometric monomial ϕ_μ. We shall need a number of properties of Z_r, beginning with the obvious representation

$$Z_r(\xi, y) = 1 + \triangle_r(\xi, y), \tag{4.8}$$

where

$$\begin{aligned} \triangle_r(\xi, y) &:= \sum_{\ell \neq 0} \left(\frac{y}{\ell + y}\right)^r e^{2\pi i \ell \xi} \\ &= y^r W_r(\xi, y), \qquad \text{if} \quad 0 \leqslant |y| < 1, \end{aligned} \tag{4.9}$$

with

$$W_r(\xi, y) := \sum_{\ell \neq 0} \frac{1}{(\ell + y)^r} e^{2\pi i \ell \xi}. \tag{4.10}$$

The properties of the function $W_r(\xi, y)$ have often been studied (see, for example, [3]). If $r \geqslant 2$ the Fourier series (4.10) is absolutely convergent, and if also $|y| \leqslant 1 - \delta$ for $\delta \in (0, 1)$ then

$$|W_r(\xi, y)| \leqslant 2 \sum_{\ell=1}^{\infty} \frac{1}{(\ell - \delta)^r} < \infty.$$

The partial derivatives of $W_r(\xi, y)$ with respect to y have a Fourier series that converge even more rapidly, so that they too are uniformly bounded for $|y| \leqslant 1 - \delta$. If $r = 1$ the Fourier series $W_1(\xi, y)$ has to be understood as the limit of the symmetric partial sums. Even in this case $W_1(\xi, y)$ is bounded independently of ξ for $|y| \leqslant 1-\delta$. To see this, note that the difference between $W_1(\xi, y)$ and $W_1(\xi, 0)$ has an absolutely convergent Fourier series with a uniformly bounded sum, while $W_1(\xi, 0)$ itself is just the Fourier series of the discontinuous periodic function defined on $(0, 1)$ by $-2\pi i(\xi - \frac{1}{2})$. Thus we have the following lemma.

LEMMA 4.1. *Let* $r \geqslant 1$. *For* $\xi \in \mathbb{R}$ *the complex-valued function* $W_r(\xi, y)$ *is* C^∞ *in* y *for* $|y| < 1$, *with* W_r *and each of its derivatives with respect to* y *being bounded uniformly in* ξ *if* $y \in [-1+\delta, 1-\delta]$, $\delta > 0$.

As a special case we have

$$|Z_r(\xi, y)| \leqslant C, \tag{4.11}$$

for some $C = C_\delta$ independent of ξ and y, if $|y| \leqslant 1-\delta$.

While the preceding lemma is well known, the next lemma appears to be new. Although easy to derive, it plays a crucial role in the arguments later in this paper, capturing an aspect of splines *not* shared by trigonometric polynomials. (Formally, the trigonometric case $\psi_\mu = \phi_\mu$ can be obtained by letting $r \to \infty$ in (4.5), in which case $Z_r(\xi, y) \to 1$. But $Z_r(\xi, y)$ replaced by 1 does *not* satisfy the property exhibited in the next lemma.)

LEMMA 4.2. *For* $\xi \in \mathbb{R}$ *and* $0 < y < 1$,

$$Z_r(\xi, y-1) = \left(\frac{y-1}{y}\right)^r e^{2\pi i\xi} Z_r(\xi, y). \tag{4.12}$$

Proof. From the definition, for $0 < y < 1$ we have

$$\begin{aligned} Z_r(\xi, y-1) &= \sum_{\ell=-\infty}^{\infty} \left(\frac{y-1}{\ell+y-1}\right)^r e^{2\pi i \ell \xi} \\ &= \left(\frac{y-1}{y}\right)^r e^{2\pi i\xi} \sum_{m=-\infty}^{\infty} \left(\frac{y}{m+y}\right)^r e^{2\pi i m\xi} \\ &= \left(\frac{y-1}{y}\right)^r e^{2\pi i\xi} Z_r(\xi, y), \end{aligned}$$

where we substituted $\ell = m+1$. ∎

Remark 3. The property (4.12) corresponds, through (4.6), to the relation

$$\psi_{\mu-n}(x) = \left(\frac{\mu-n}{\mu}\right)^r \psi_\mu(x), \qquad \mu \not\equiv 0,$$

which is a trivial consequence of (4.12) with an appropriately extended domain of definition of ψ_μ.

Finally, the following simple lemma is often useful when working with periodic splines.

LEMMA 4.3. *Let $\alpha > 1$ and $\mu \in \Lambda_h^*$. There exist constants C_1 and C_2 independent of μ such that*

$$\text{(a)} \qquad \sum_{\substack{m \equiv \mu, \\ m \neq \mu}} \left| \frac{\mu}{m} \right|^\alpha \leqslant C_1 \left| \frac{\mu}{n} \right|^\alpha, \tag{4.13}$$

$$\text{(b)} \qquad \sum_{m \equiv \mu} \left| \frac{\mu}{m} \right|^\alpha \leqslant C_2. \tag{4.14}$$

Proof. Putting $m = \ell n + \mu$, we obtain

$$\sum_{\substack{m \equiv \mu, \\ m \neq \mu}} \left| \frac{\mu}{m} \right|^\alpha = \sum_{\ell \neq 0} \left| \frac{\mu}{\ell n + \mu} \right|^\alpha = \left| \frac{\mu}{n} \right|^\alpha \sum_{\ell \neq 0} \frac{1}{|\ell + \mu/n|^\alpha}$$

$$\leqslant \left| \frac{\mu}{n} \right|^2 2 \sum_{\ell = 1}^{\infty} \frac{1}{(\ell - 1/2)^\alpha},$$

because $|\mu| \leqslant n/2$. For the same reason

$$\sum_{m \equiv \mu} \left| \frac{\mu}{m} \right|^\alpha = 1 + \sum_{\substack{m \equiv \mu, \\ m \neq \mu}} \left| \frac{\mu}{m} \right|^\alpha \leqslant 1 + \frac{1}{2^{\alpha - 1}} \sum_{\ell = 1}^{\infty} \frac{1}{(\ell - 1/2)^\alpha} = C_2. \quad \blacksquare$$

5. PROPERTIES OF DISCRETE ORTHOGONAL PROJECTION R_h

From the defining equation (2.7) for the operator R_h we see that

$$R_h v = \sum_{\mu \in \Lambda_h} \frac{(v, \psi'_\mu)_h}{(\psi'_\mu, \psi'_\mu)_h} \psi'_\mu,$$

where we have used a property of the basis $\{\psi'_\mu\}$ following from (4.3), namely that $(\psi'_\mu, \psi'_\nu)_h = 0$ if $\mu \not\equiv \nu$.

With the aid of formula (4.6) (with r replaced by r'), it then follows that

$$R_h v = \sum_{m \equiv 0} \hat{v}(m) \sum_{j=1}^{J} \omega_j \phi_{m/n}(\xi_j)$$
$$+ \sum_{\mu \in \Lambda_h^*} D_0^{-1}\left(\frac{\mu}{n}\right) \sum_{m \equiv \mu} \hat{v}(m) \sum_{j=1}^{J} \omega_j \phi_{(m-\mu)/n}(\xi_j) \overline{Z_{r'}\left(\xi_j, \frac{\mu}{n}\right)} \psi'_\mu, \tag{5.1}$$

where

$$D_0(y) = \sum_j \omega_j \, |Z_{r'}(\xi_j, y)|^2, \qquad |y| < 1. \tag{5.2}$$

The expression (5.1) is a special case of [4, Eq. (3.2)]: Indeed in the language of [4], $R_h v$ defined by (2.7) is the "qualocation" approximation to v for the special case in which the operator in the defining equation for v is the identity.

Lemma 5.1. *Under the assumption that $J \geqslant 2$, or if $J = 1$ that $\xi_1 \neq \frac{1}{2}$ if r' is even, and that $\xi_1 \neq 0$ if r' is odd, $D_0(y)$ is positive and continuous on $(-1, 1)$.*

Proof. The continuity of D_0 follows from (5.2), (4.8), (4.9), and Lemma 4.1, while the strict positivity follows from a result established in [3], that $Z_{r'}(\xi, y)$ vanishes on $[0, 1) \times (0, 1)$ only if $\xi = \frac{1}{2}$ and $y = \frac{1}{2}$ for r' even, and only if $\xi = 0$ and $y = \frac{1}{2}$ for r' odd. ▮

Further useful properties of the function D_0 are expressed in:

Lemma 5.2. *The function D_0 defined by (5.2) is real, even, and C^∞ for $y \in (-1, 1)$. Moreover it has the representation*

$$D_0(y) = 1 + y^{r'} A(y), \qquad |y| < 1, \tag{5.3}$$

where A is C^∞ on $(-1, 1)$.

Proof. That D_0 is real is apparent from (5.2), and that it is even follows from (5.2) and (4.7) (with r replaced by r'). The C^∞ nature of D_0 and the representation (5.3) follow from the definition (5.2) together with (4.8), (4.9) and Lemma 4.1. ▮

The following approximation property for R_h is a special case of [4, Theorem 2].

LEMMA 5.3. *Under the assumptions of Lemma* 5.1, *for all* σ, τ *satisfying*

$$\sigma < r' - \tfrac{1}{2}, \qquad \tau > \tfrac{1}{2}, \qquad 0 \leqslant \sigma \leqslant \tau \leqslant r'$$

there exists $C > 0$ *such that*

$$\|R_h v - v\|_\sigma \leqslant C h^{\tau - \sigma} \|v\|_\tau, \tag{5.4}$$

if $v \in H^\tau$.

It may be useful to remark here that (in contrast to the thrust of [4]) we are here making *no* assumption on the quadrature rule Q, beyond eliminating the known unstable rules, and requiring positive weights ω_j. For that reason it would be unreasonable to expect any "higher-order" or "negative-norm" approximation properties of R_h.

In the limit $r' \to \infty$ we see from (5.3) that $D_0(y) \to 1$, which is the approximate result if the spline space $S_h^{r'}$ is replaced by the corresponding space of trigonometric polynomials. The following relation, holding for any finite r', is therefore one that distinguishes the spline spaces from the trigonometric polynomial spaces. This lemma, and the one that follows it, will be useful to us in subsequent sections.

LEMMA 5.4. *For* $0 < y < 1$,

$$D_0(y-1) = \left(\frac{y-1}{y}\right)^{2r'} D_0(y).$$

Proof. With the aid of (5.2) and Lemma 4.2 we have

$$\begin{aligned} D_0(y-1) &= \sum_j \omega_j \, |Z_{r'}(\xi_j, y-1)|^2 \\ &= \left(\frac{y-1}{y}\right)^{2r'} \sum_j \omega_j \, |Z_{r'}(\xi_j, y)|^2 \\ &= \left(\frac{y-1}{y}\right)^{2r'} D_0(y). \quad \blacksquare \end{aligned}$$

LEMMA 5.5. *For* $0 < y < 1$ *and* $\xi \in \mathbb{R}$,

$$D_0^{-1}(y-1)\, Z_{r'}(\xi, y-1)(y-1)^{r'} = D_0^{-1}(y)\, e^{2\pi i \xi} Z_{r'}(\xi, y)\, y^{r'}.$$

Proof. This is an immediate consequence of Lemmata 4.2 and 5.4. ∎

6. PROOF OF THEOREM 2.1

In the following proof we assume, in addition to the conditions (2.9), that $s \leqslant t$. The result can then be shown to hold if t is replaced by any smaller value, say t', by appeal to the standard inverse inequality

$$\|v_h\|_t \leqslant Ch^{t'-t} \|v_h\|_{t'} \qquad \text{for} \quad v_h \in S_h, \quad t' < t.$$

From definition (2.8) of the operator L, we may write

$$L = (a+b)J + aL_+ + bL_-, \tag{6.1}$$

where $a, b \in \mathbb{R}$,

$$Jv(x) := \hat{v}(0), \tag{6.2}$$

and

$$L_\pm v(x) := \sum_{m \neq 0} [m]_\beta \, e^{2\pi imx} \hat{v}(m), \tag{6.3}$$

with

$$[m]_\beta := \begin{cases} |m|^\beta & \text{in the + case,} \\ (\text{sign } m)\, |m|^\beta & \text{in the } - \text{ case.} \end{cases} \tag{6.4}$$

Clearly, it is enough to prove estimate (2.10) in the theorem for the three special cases $L = J$, $L = L_+$, and $L = L_-$.

To prove (2.10) for case $L = J$, note that in this case Jv_h is a constant, and hence $R_h J v_h = J v_h$ and $\|Jv_h\|_{s-\beta} = |\hat{v}_h(0)| \leqslant \|v_h\|_t$, so we only have to show

$$\|R_h f - f\|_{s-\beta} \leqslant Ch^{t-s+\delta} \|f\|_\kappa. \tag{6.5}$$

If $s < \beta$ we can obtain (6.5) by arguing, with the aid of Lemma 5.3, that

$$\|R_h f - f\|_{s-\beta} \leqslant \|R_h f - f\|_0 \leqslant Ch^{r'} \|f\|_{r'} \leqslant Ch^{t-s+\delta} \|f\|_{r'},$$

since by assumption (2.9) we have $t - s + \delta \leqslant r'$. On the other hand, if $s \geqslant \beta$ we have, again using Lemma 5.3,

$$\|R_h f - f\|_{s-\beta} \leqslant Ch^{r'-s+\beta} \|f\|_{r'} \leqslant Ch^{t-s+\delta} \|f\|_{r'},$$

where the second inequality follows because by (2.9) $t \leqslant r' + \beta - \delta$. Thus, noting that $r' < \kappa$, we see that (6.5) holds for all s and β.

It remains to prove (2.10) for the special cases $L = L_\pm$. That is our task in the remainder of this section.

According to (4.4), we may write an arbitrary spline function $v_h \in S_h$ in the form

$$v_h = \sum_{\mu \in \Lambda_h} \hat{v}_h(\mu)\, \psi_\mu,$$

thus

$$L_\pm v_h = \sum_{\mu \in \Lambda_h^*} \hat{v}_h(\mu)\, L_\pm \psi_\mu,$$

and from (4.3) and (6.3) we obtain

$$L_\pm v_h = \sum_{\mu \in \Lambda_h^*} \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \phi_m. \tag{6.6}$$

With the aid of (5.1), this gives

$$R_h L_\pm v_h = \sum_{\mu \in \Lambda_h^*} D_0^{-1}\left(\frac{\mu}{n}\right) \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \times \sum_{j=1}^{J} \omega_j \phi_{(m-\mu)/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{\mu}{n}\right)}\, \psi'_\mu.$$

Finally, we multiply by

$$f = \sum_{a \in \mathbb{Z}} \hat{f}(a)\, \phi_a, \tag{6.7}$$

and use the Fourier expansion of ψ'_μ, to obtain

$$fR_h L_\pm v_h = \sum_{a \in \mathbb{Z}} \hat{f}(a) \sum_{\mu \in \Lambda_h^*} D_0^{-1}\left(\frac{\mu}{n}\right) \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \times \sum_{j=1}^{J} \omega_j \phi_{(m-\mu)/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{\mu}{n}\right)} \sum_{\ell \equiv \mu + a} \left(\frac{\mu}{\ell - a}\right)^{r'} \phi_\ell. \tag{6.8}$$

On the other hand, from (6.6) and (6.7) we have

$$fL_\pm v_h = \sum_{a \in \mathbb{Z}} \hat{f}(a) \sum_{\mu \in \Lambda_h^*} \hat{v}_h(\mu) \sum_{\ell \equiv \mu + a} [\ell - a]_\beta \left(\frac{\mu}{\ell - a}\right)^r \phi_\ell. \tag{6.9}$$

To permit the easy application of (5.1), it is convenient to replace the sum over μ in this equation by a sum over $v := (\mu + a)(n)$, where $m(n)$ denotes the unique integer in Λ_h that differs from m by a multiple of n; i.e., $m(n)$ (which we read as "m mod n") is defined by

$$m(n) \in \Lambda_h, \qquad m(n) \equiv m.$$

For $\mu, v \in \Lambda_h$ we note that

$$v = (\mu + a)(n) \Leftrightarrow \mu = (v - a)(n), \tag{6.10}$$

from which it follows that (6.9) may be rewritten as

$$fL_{\pm} v_h = \sum_{a \in \mathbb{Z}} \hat{f}(a) \sum_{\substack{v \in \Lambda_h, \\ v \not\equiv a}} \hat{v}_h((v-a)(n)) \sum_{\ell \equiv v} [\ell - a]_\beta \left(\frac{(v-a)(n)}{\ell - a} \right)^r \phi_\ell.$$

Finally, (5.1) gives

$$R_h fL_{\pm} v_h = \sum_{\substack{a \in \mathbb{Z} \\ a \not\equiv 0}} \hat{f}(a)\, \hat{v}_h((-a)(n)) \sum_{\ell \equiv 0} [\ell - a]_\beta \left(\frac{(-a)(n)}{\ell - a} \right)^r \sum_j \omega_j \phi_{\ell/n}(\xi_j)$$

$$+ \sum_{a \in \mathbb{Z}} \hat{f}(a)$$

$$\times \sum_{\substack{v \in \Lambda_h^*, \\ v \not\equiv a}} D_0^{-1}\left(\frac{v}{n}\right) \hat{v}_h((v-a)(n)) \sum_{\ell \equiv v} [\ell - a]_\beta \left(\frac{(v-a)(n)}{\ell - a} \right)^r$$

$$\times \sum_j \omega_j \phi_{(\ell - v)/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{v}{n}\right)}\, \psi'_v,$$

or using again (6.10), and inserting the Fourier representation for ψ'_v,

$$R_h fL_{\pm} v_h = \sum_{\substack{a \in \mathbb{Z} \\ a \not\equiv 0}} \hat{f}(a)\, \hat{v}_h((-a)(n)) \sum_{m \equiv -a} [m]_\beta \left(\frac{(-a)(n)}{m} \right)^r \sum_j \omega_j \phi_{(m+a)/n}(\xi_j)$$

$$+ \sum_{a \in \mathbb{Z}} \hat{f}(a) \sum_{\substack{\mu \in \Lambda_h^* \\ \mu \not\equiv -a}} D_0^{-1}\left(\frac{(\mu + a)(n)}{n} \right) \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m} \right)^r$$

$$\times \sum_j \omega_j \phi_{(m + a - (\mu + a)(n))/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{(\mu + a)(n)}{n}\right)}$$

$$\times \sum_{\ell \equiv \mu + a} \left(\frac{(\mu + a)(n)}{\ell} \right)^{r'} \phi_\ell. \tag{6.11}$$

Our task is to compute appropriate Sobolev norms of the difference

$$T := R_h fL_{\pm} v_h - fR_h L_{\pm} v_h; \tag{6.12}$$

specifically, we have to show that $\|T\|_{s-\beta}$ satisfies estimate (2.10). It is convenient to split T into two parts U and Y, the first containing the "small" values of a, which we define to be those with $|a| \leqslant n/4$, and a remainder Y, containing the contributions to T from $|a| > n/4$. Thus

$$T = U + Y. \tag{6.13}$$

The term U we split further: We write

$$U := U_1 + U_2, \tag{6.14}$$

where U_1 contains the terms involving $\hat{v}_h(\mu)$ with $\mu + a \not\equiv 0$, and of course $|a| \leqslant n/4$, while U_2 contains the terms involving $\hat{v}_h(\mu)$ with $\mu + a \equiv 0$ and $|a| \leqslant n/4$. Thus

$$\begin{aligned} U_1 = \sum_{|a| \leqslant n/4} \hat{f}(a) \sum_{\substack{\mu \in \Lambda_h^* \\ \mu \not\equiv -a}} \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \sum_j \omega_j \\ \times \sum_{\ell \equiv \mu + a} \Bigg[D_0^{-1} \left(\frac{(\mu+a)(n)}{n}\right) \phi_{(m+a-(\mu+a)(n))/n}(\xi_j) \\ \times \overline{Z_{r'}\left(\xi_j, \frac{(\mu+a)(n)}{n}\right) \left(\frac{(\mu+a)(n)}{\ell}\right)^{r'}} \\ - D_0^{-1}\left(\frac{\mu}{n}\right) \phi_{(m-\mu)/n}(\xi_j) \overline{Z_{r'}\left(\xi_j, \frac{\mu}{n}\right) \left(\frac{\mu}{\ell - a}\right)^{r'}} \Bigg] \phi_\ell. \end{aligned} \tag{6.15}$$

Fortunately, this expression can be simplified by appealing to Lemma A1 in the appendix, which gives

$$\begin{aligned} D_0^{-1} \left(\frac{(\mu+a)(n)}{n}\right) \phi_{(m+a-(\mu+a)(n))/n}(\xi) \overline{Z_{r'}\left(\xi, \frac{(\mu+a)(n)}{n}\right) \left(\frac{(\mu+a)(n)}{\ell}\right)^{r'}} \\ = D_0^{-1}\left(\frac{\mu+a}{n}\right) \phi_{(m-\mu)/n}(\xi) \overline{Z_{r'}\left(\xi, \frac{\mu+a}{n}\right)\left(\frac{\mu+a}{\ell}\right)^{r'}}, \end{aligned} \tag{6.16}$$

in effect allowing the awkward quantity $(\mu + a)(n)$ in (6.15) to be replaced by $\mu + a$. (It is at this crucial stage that the argument fails if the spline test space is replaced by a space of trigonometric polynomials.) With this substitution we have

$$U_1 = \sum_{|a| \leqslant n/4} V_a, \tag{6.17}$$

where, for $|a| \leqslant n/4$,

$$V_a := \hat{f}(a) \sum_{\substack{\mu \in \Lambda_h^*, \\ \mu \not\equiv -a}} \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \sum_j \omega_j \phi_{(m-\mu)/n}(\xi_j)$$
$$\times \sum_{\ell \equiv \mu + a} E(n, \mu, a, \ell, \xi_j) \phi_\ell,$$

with, for $\ell \equiv \mu + a$, $\mu \neq 0$, and $\mu \not\equiv -a$,

$$E(n, \mu, a, \ell, \xi) := D_0^{-1}\left(\frac{\mu+a}{n}\right) \overline{Z_{r'}\left(\xi, \frac{\mu+a}{n}\right)} \left(\frac{\mu+a}{\ell}\right)^{r'}$$
$$- D_0^{-1}\left(\frac{\mu}{n}\right) \overline{Z_{r'}\left(\xi, \frac{\mu}{n}\right)} \left(\frac{\mu}{\ell - a}\right)^{r'}. \tag{6.18}$$

From Definition (2.1) of the Sobolev norm, and the bound on $E(n, \mu, a, \ell, \xi)$ given by Lemma A2 in the appendix, we now obtain

$$\|V_a\|_{s-\beta}^2 = |\hat{f}(a)|^2 \sum_{\substack{\mu \in \Lambda_h^*, \\ \mu \not\equiv -a}} \sum_{\ell \equiv \mu + a} |\ell|^{2(s-\beta)} \left| \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \right.$$
$$\left. \times \sum_j \omega_j \phi_{(m-\mu)/n}(\xi_j)\, E(n, \mu, a, \ell, \xi_j) \right|^2$$
$$\leqslant |\hat{f}(a)|^2 \sum_{\substack{\mu \in \Lambda_h^*, \\ \mu \not\equiv -a}} |\hat{v}_h(\mu)|^2 \left(\sum_{m \equiv \mu} |m|^{\beta - r} |\mu|^r \right)^2$$
$$\times \sum_{\ell \equiv \mu + a} |\ell|^{2(s-\beta)} \sup_\xi |E(n, \mu, a, \ell, \xi)|^2$$
$$\leqslant C\, |\hat{f}(a)|^2 \sum_{\substack{\mu \in \Lambda_h^*, \\ \mu \not\equiv -a}} |\hat{v}_h(\mu)|^2\, |\mu|^{2\beta} \left(|a|\, |\mu|^{r'-1} + |a|^{r'}\right)^2$$
$$\times \left[|\mu + a|^{2(s-\beta)}\, n^{-2r'} + \sum_{\substack{\ell \equiv \mu + a, \\ \ell \neq \mu + a}} |\ell|^{2(s-\beta-r')} \right], \tag{6.19}$$

where in the last step we used also Lemma 4.3(b).

In (6.19) we now make use of the elementary bounds for integers a and μ, with $\mu \neq 0$ and $\mu + a \neq 0$,

$$|a|\;|\mu|^{r'-1} + |a|^{r'} \leqslant 2\;|a|^{r'}\;|\mu|^{r'-1},$$

and

$$|\mu + a|^{s-\beta} \leqslant C \max(1, |a|)^{\max(s-\beta,0)}\;|\mu|^{\max(s-\beta,0)},$$

together with the bound in Lemma A4, with the parameter α in that lemma set equal to $2(r'+\beta-s)$. Thus we obtain from (6.19)

$$\|V_a\|_{s-\beta}^2 \leqslant C\,|\hat{f}(a)|^2 \max(1, |a|)^{2(r'+\max(s-\beta,0))} \sum_{\mu\in\Lambda_h^*} |\hat{v}_h(\mu)|^2 \times (|\mu|^{2(r'-1+\max(s,\beta))} n^{-2r'} + |\mu|^{2(r'-1+\beta)} n^{2(s-\beta-r')}). \tag{6.20}$$

From this estimate and Lemma A5 we obtain

$$\|V_a\|_{s-\beta}^2 \leqslant C\,|\hat{f}(a)|^2 \max(1, |a|)^{2(r'+\max(s-\beta,0))}\;\|v_h\|_t^2 \times (n^{2(-r'+\max(r'-1+\max(s,\beta)-t,0))} + n^{2(-r'-\beta+s+\max(r'-1+\beta-t,0))}),$$

or

$$\begin{aligned}\|V_a\|_{s-\beta} &\leqslant C\,|\hat{f}(a)| \max(1, |a|)^{r'+\max(s-\beta,0)}\;\|v_h\|_t\, h^{\min(t-s+1,\,t-\beta+1,\,r',\,r'+\beta-s)} \\ &\leqslant Ch^{t-s+\delta}\,|\hat{f}(a)| \max(1, |a|)^{r'+\max(s-\beta,0)}\;\|v_h\|_t,\end{aligned} \tag{6.21}$$

because we have $\delta \leqslant 1$, $s \geqslant \beta - 1 + \delta$, $t - s \leqslant r' - \delta$, $t \leqslant r' + \beta - \delta$.

Finally, from (6.17) we have

$$\begin{aligned}\|U_1\|_{s-\beta} &\leqslant \sum_{|a|\leqslant n/4} \|V_a\|_{s-\beta} \\ &\leqslant Ch^{t-s+\delta} \sum_{|a|\leqslant n/4} |\hat{f}(a)| \max(1, |a|)^{r'+\max(s-\beta,0)+\nu} \\ &\quad \times \max(1, |a|)^{-\nu}\;\|v_h\|_t \\ &\leqslant Ch^{t-s+\delta} \left(\sum_{|a|\leqslant n/4} |\hat{f}(a)|^2 \max(1, |a|)^{2(r'+\max(s-\beta,0)+\nu)} \right)^{1/2} \\ &\quad \times \left(\sum_{|a|\leqslant n/4} \max(1, |a|)^{-2\nu} \right)^{1/2} \|v_h\|_t \\ &\leqslant Ch^{t-s+\delta}\;\|f\|_{r'+\max(s-\beta,0)+\nu}\;\|v_h\|_t,\end{aligned}$$

because $\nu > \frac{1}{2}$. Thus for the term U_1 the bound in the theorem is satisfied.

The argument for the term U_2 in (6.14) is similar but easier. From (6.11) and (6.8) we have $U_2=\sum_{|a|\leqslant n/4,\, a\neq 0} W_a$, where, for $|a|\leqslant n/4$ and $a\neq 0$,

$$W_a=\hat{f}(a)\,\hat{v}_h(-a)\sum_{m\equiv -a}[m]_\beta\left(-\frac{a}{m}\right)^r\sum_j \omega_j\phi_{(m+a)/n}(\xi_j)$$
$$\times\sum_{\ell\equiv 0}K(n,a,\ell,\xi_j)\,\phi_\ell,$$

and

$$K(n,a,\ell,\xi)=\delta_{\ell 0}-D_0^{-1}\left(-\frac{a}{n}\right)\overline{Z_{r'}\left(\xi,-\frac{a}{n}\right)}\left(-\frac{a}{\ell-a}\right)^{r'}.$$

By appeal to Lemma A3 we can show

$$\|W_a\|_{s-\beta}^2\leqslant C\,|\hat{f}(a)|^2\,|\hat{v}_h(-a)|^2$$
$$\times\left(\sum_{m\equiv -a}|m|^{\beta-r}\,|a|^r\right)^2\left(\left|\frac{a}{n}\right|^{2r'}+\sum_{\substack{\ell\equiv 0,\\ \ell\neq 0}}|\ell|^{2(s-\beta)}\left|\frac{a}{\ell}\right|^{2r'}\right)$$
$$\leqslant C\,|\hat{f}(a)|^2\,|\hat{v}_h(-a)|^2\,|a|^{2(\beta+r')}\,(n^{-2r'}+n^{-2(r'+\beta-s)}),$$

where the last step follows from Lemma 4.3b and the assumption that $s<r'+\beta-\frac{1}{2}$. Thus

$$\|W_a\|_{s-\beta}\leqslant Ch^{\min(r',\,r'+\beta-s)}\,|\hat{f}(a)|\,|a|^{\beta+r'-t}\,\|v_h\|_t.$$

Now because $a\neq 0$ and $|a|\leqslant n/4$,

$$|a|^{\beta+r'-t}\leqslant|a|^{r'+\max(\beta-t,\,0)}\leqslant n^{\max(\beta-t,\,0)}\,|a|^{r'}=h^{\min(t-\beta,\,0)}\,|a|^{r'},$$

so that

$$\|W_a\|_{s-\beta}\leqslant Ch^{\min(t-s+r',\,t-\beta+r',\,r',\,r'+\beta-s)}\,|\hat{f}(a)|\,|a|^{r'}\,\|v_h\|_t$$
$$\leqslant Ch^{t-s+\delta}\,|\hat{f}(a)|\,|a|^{r'}\,\|v_h\|_t.$$

Since this is analogous to the bound in (6.21), it follows as before that

$$\|U_2\|_{s-\beta}\leqslant Ch^{t-s+\delta}\,\|f\|_{r'+\nu}\,\|v_h\|_t,$$

and therefore the bound in the theorem is satisfied by the term U_2.

The "remainder" term Y, i.e., the contribution to the difference of (6.11) and (6.8) from the terms with $|a|>n/4$, may be written as

$$Y=\sum_{|a|>n/4}Y_a=\sum_{|a|>n/4}(Y_{1,a}+Y_{2,a}+Y_{3,a}),\tag{6.22}$$

where, for $|a| > n/4$, $Y_{1,a}$ comes from the second term of (6.11),

$$Y_{1,a} := \hat{f}(a) \sum_{\substack{\mu \in \Lambda_h^*, \\ \mu \not\equiv -a}} D_0^{-1}\left(\frac{(\mu+a)(n)}{n}\right) \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r$$
$$\times \sum_j \omega_j \phi_{(m+a-(\mu+a)(n))/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{(\mu+a)(n)}{n}\right)}$$
$$\times \sum_{\ell \equiv \mu+a} \left(\frac{(\mu+a)(n)}{\ell}\right)^{r'} \phi_\ell,$$

so that

$$\| Y_{1,a} \|_{s-\beta}^2 = |\hat{f}(a)|^2 \sum_{\substack{\mu \in \Lambda_h^* \\ \mu \not\equiv -a}} \left| D_0^{-1}\left(\frac{(\mu+a)(n)}{n}\right)\right|^2 |\hat{v}_h(\mu)|^2 \left| \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r \right.$$
$$\left. \times \sum_j \omega_j \phi_{(m+a-(\mu+a)(n))/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{(\mu+a)(n)}{n}\right)}\right|^2$$
$$\times \sum_{\ell \equiv \mu+a} |\ell|^{2(s-\beta)} \left| \frac{(\mu+a)(n)}{\ell} \right|^{2r'}$$
$$\leqslant C\, |\hat{f}(a)|^2 \sum_{\substack{\mu \in \Lambda_h^* \\ \mu \not\equiv -a}} |\hat{v}_h(\mu)|^2 \left(\sum_{m \equiv \mu} |m|^{\beta - r}\, |\mu|^r \right)^2 |(\mu+a)(n)|^{2(s-\beta)}$$
$$\leqslant C\, |\hat{f}(a)|^2 \sum_{\mu \in \Lambda_h^*} |\hat{v}_h(\mu)|^2\, |\mu|^{2\beta}\, n^{2\max(s-\beta, 0)},$$

where we have used (4.11), Lemmata 5.1 and 4.3(b), noting that $r' - s + \beta > \frac{1}{2}$ and $r > \beta + 1$. With the aid of Lemma A5, we then obtain

$$\| Y_{1,a} \|_{s-\beta} \leqslant C h^{\min(t-\beta,\, \beta - s,\, 0)}\, |\hat{f}(a)|\, \| v_h \|_t, \tag{6.23}$$

where we used the temporary assumption $s \leqslant t$ made at the start of this section to simplify the exponent of h. Next, $Y_{2,a}$ is the contribution to Y_a from (6.8),

$$Y_{2,a} := -\hat{f}(a) \sum_{\mu \in \Lambda_h^*} D_0^{-1}\left(\frac{\mu}{n}\right) \hat{v}_h(\mu) \sum_{m \equiv \mu} [m]_\beta \left(\frac{\mu}{m}\right)^r$$
$$\times \sum_j \omega_j \phi_{(m-\mu)/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{\mu}{n}\right)} \sum_{\ell \equiv \mu+a} \left(\frac{\mu}{\ell - a}\right)^{r'} \phi_\ell,$$

and hence

$$\begin{aligned}\|Y_{2,a}\|_{s-\beta}^2 &= |\hat{f}(a)|^2 \sum_{\mu\in\Lambda_h^*} \left|D_0^{-1}\left(\frac{\mu}{n}\right)\right|^2 |\hat{v}_h(\mu)|^2 \left|\sum_{m\equiv\mu} [m]_\beta \left(\frac{\mu}{m}\right)^r\right. \\ &\quad \left.\times \sum_j \omega_j \phi_{(m-\mu)/n}(\xi_j)\, \overline{Z_{r'}\left(\xi_j, \frac{\mu}{n}\right)}\right|^2 \\ &\quad \times \sum_{\ell\equiv\mu+a} \max(1, |\ell|)^{2(s-\beta)} \left|\frac{\mu}{\ell-a}\right|^{2r'} \\ &\leqslant C\,|\hat{f}(a)|^2 \sum_{\mu\in\Lambda_h^*} |\hat{v}_h(\mu)|^2 \left(\sum_{m\equiv\mu} |m|^{\beta-r}\,|\mu|^r\right)^2 \\ &\quad \times \sum_{\ell\equiv\mu+a} \max(1, |\ell|)^{2(s-\beta)} \left|\frac{\mu}{\ell-a}\right|^{2r'}.\end{aligned}$$

Now, for $\mu\in\Lambda_h^*$,

$$\begin{aligned}&\sum_{\ell\equiv\mu+a} \max(1, |\ell|)^{2(s-\beta)} \left|\frac{\mu}{\ell-a}\right|^{2r'} \\ &\quad = |\mu|^{2r'} \sum_{p\equiv\mu} \frac{\max(1, |p+a|)^{2(s-\beta)}}{|p|^{2r'}} \\ &\quad \leqslant C\,|\mu|^{2r'} \sum_{p\equiv\mu} \frac{|p|^{2\max(s-\beta,0)} + |a|^{2\max(s-\beta,0)}}{|p|^{2r'}} \\ &\quad \leqslant C(|\mu|^{2\max(s-\beta,0)} + |a|^{2\max(s-\beta,0)}),\end{aligned}$$

where we used Lemma 4.3(b) (twice) and $s < r' + \beta - \frac{1}{2}$. Thus

$$\|Y_{2,a}\|_{s-\beta}^2 \leqslant C\,|\hat{f}(a)|^2 \sum_{\mu\in\Lambda_h^*} |\hat{v}_h(\mu)|^2\,(|\mu|^{2\max(s,\beta)} + |\mu|^{2\beta}\,|a|^{2\max(s-\beta,0)}),$$

and Lemma A5 now gives

$$\begin{aligned}\|Y_{2,a}\|_{s-\beta}^2 &\leqslant C(n^{2\max(\max(s,\beta)-t,0)} \\ &\quad + n^{2\max(\beta-t,0)}\,|a|^{2\max(s-\beta,0)})\,|\hat{f}(a)|^2\,\|v_h\|_t.\end{aligned}$$

Thus we may write (using again $t \geqslant s$)

$$\|Y_{2,a}\|_{s-\beta} \leqslant Ch^{\min(t-\beta,0)}\,|a|^{\max(s-\beta,0)}\,|\hat{f}(a)|\,\|v_h\|_t. \tag{6.24}$$

Finally, $Y_{3,a}$ is the contribution from the first term of (6.11), thus for $a \not\equiv 0$

$$Y_{3,a} := \hat{f}(a)\, \hat{v}_h((-a)(n)) \sum_{m \equiv -a} [m]_\beta \left(\frac{(-a)(n)}{m}\right)^r \sum_j \omega_j \phi_{(m+a)/n}(\xi_j),$$

and hence

$$\begin{aligned} \| Y_{3,a} \|_{s-\beta} &= \left| \hat{f}(a)\, \hat{v}_h((-a)(n)) \sum_{m \equiv -a} [m]_\beta \left(\frac{(-a)(n)}{m}\right)^r \sum_j \omega_j \phi_{(m+a)/n}(\xi_j) \right| \\ &\leqslant |\hat{f}(a)|\, |\hat{v}_h((-a)(n))| \sum_{m \equiv -a} |m|^\beta \left| \frac{(-a)(n)}{m} \right|^r \\ &\leqslant C\, |\hat{f}(a)|\, |\hat{v}_h((-a)(n))|\, |(-a)(n)|^\beta \\ &\leqslant C n^{\max(\beta - t, 0)}\, |\hat{f}(a)|\, |\hat{v}_h((-a)(n))|\, |(-a)(n)|^t \\ &\leqslant C h^{\min(t - \beta, 0)}\, |\hat{f}(a)|\, \| v_h \|_t. \end{aligned} \tag{6.25}$$

For $a \equiv 0$ we define $Y_{3,a} := 0$.

Now (6.22)–(6.25) give

$$\begin{aligned} \| Y \|_{s-\beta} &\leqslant \sum_{|a| > n/4} (\| Y_{1,a} \|_{s-\beta} + \| Y_{2,a} \|_{s-\beta} + \| Y_{3,a} \|_{s-\beta}) \\ &\leqslant C \sum_{|a| > n/4} (h^{\min(t-\beta, \beta - s, 0)} + h^{\min(t-\beta, 0)}\, |a|^{\max(s-\beta, 0)})\, |\hat{f}(a)|\, \| v_h \|_t. \end{aligned}$$

Because $|a| > n/4$ and therefore $h\,|a| > 1/4$, in the first term we may use

$$1 \leqslant C(h\,|a|)^{\max(t - s + \delta - \min(t-\beta, \beta - s, 0), 0)},$$

so that

$$h^{\min(t-\beta, \beta - s, 0)} \leqslant C h^{t-s+\delta}\, |a|^{\max(\beta - s + \delta, t - \beta + \delta, t - s + \delta, 0)},$$

where we used the fact that h raised to a positive power is bounded above by 1; and in the second term

$$1 \leqslant C(h|a|)^{\max(t - s + \delta - \min(t - \beta, 0), 0)},$$

so that

$$h^{\min(t-\beta, 0)}\, |a|^{\max(s-\beta, 0)} \leqslant C h^{t-s+\delta}\, |a|^{\max(\beta - s + \delta, t - \beta + \delta, t - s + \delta, s - \beta, \delta, 0)}.$$

Thus altogether we have

$$\begin{aligned}\|Y\|_{s-\beta} &\leqslant Ch^{t-s+\delta} \sum_{|a|>n/4} |a|^{\max(\beta-s+\delta,\, t-\beta+\delta,\, t-s+\delta,\, s-\beta,\, \delta,\, 0)} \, |\hat{f}(a)| \, \|v_h\|_t \\ &\leqslant Ch^{t-s+\delta} \, \|f\|_{\max(\beta-s+\delta+v,\, t-\beta+\delta+v,\, t-s+\delta+v,\, s-\beta+v,\, \delta+v,\, v)} \, \|v_h\|_t \\ &\leqslant Ch^{t-s+\delta} \, \|f\|_{r'+v} \, \|v_h\|_t,\end{aligned}$$

where the last step follows because, by (2.9), $t-s+\delta \leqslant r'$, $t-\beta+\delta \leqslant r'$, $\beta-s+\delta \leqslant 1$, $s-\beta < r'-\frac{1}{2}$, and $\delta \leqslant 1$. Thus the term Y in (6.13) satisfies the bound in (2.10), and the proof of the theorem is complete. ∎

A. APPENDIX: MISCELLANEOUS LEMMATA

The lemmata in this appendix are used in the proof of the main theorem, Theorem 2.1.

LEMMA A1. *Assume $\mu \in \Lambda_h$, $|a| \leqslant n/4$, $\xi \in \mathbb{R}$, $\phi_\alpha(x) = e^{2\pi i\alpha x}$, $Z_r(\xi, y)$ is as defined by* (4.7), *and $D_0(y)$ is as defined by* (5.2). *Then*

$$\begin{aligned}&D_0^{-1}\left(\frac{(\mu+a)(n)}{n}\right) \phi_{-(\mu+a)(n)/n}(\xi)\, \overline{Z_{r'}\left(\xi, \frac{(\mu+a)(n)}{n}\right)}\, ((\mu+a)(n))^{r'} \\ &\quad = D_0^{-1}\left(\frac{\mu+a}{n}\right) \phi_{-(\mu+a)/n}(\xi)\, \overline{Z_{r'}\left(\xi, \frac{\mu+a}{n}\right)}\, (\mu+a)^{r'}.\end{aligned}$$

Proof. The result is immediate if $(\mu+a)(n) = \mu+a$. Suppose $(\mu+a)(n) = \mu+a-n$. Then the complex conjugate of Lemma 5.5 may be applied with $y = (\mu+a)/n$, noting that $\frac{1}{2} < y \leqslant \frac{3}{4}$. Alternatively, suppose $(\mu+a)(n) = \mu+a+n$. Then the conjugate of Lemma 5.5 may be applied with $y = (\mu+a+n)/n$. Since all possibilities are thereby exhausted, the result is proved. ∎

LEMMA A2. *Let*

$$\begin{aligned}E(n, \mu, a, \ell, \xi) := D_0^{-1}\left(\frac{\mu+a}{n}\right) \overline{Z_{r'}\left(\xi, \frac{\mu+a}{n}\right)} \left(\frac{\mu+a}{\ell}\right)^{r'} \\ - D_0^{-1}\left(\frac{\mu}{n}\right) \overline{Z_{r'}\left(\xi, \frac{\mu}{n}\right)} \left(\frac{\mu}{\ell-a}\right)^{r'},\end{aligned}$$

where $\mu \in \Lambda_h^*$, $|a| \leqslant n/4$, $\mu \not\equiv -a$, *and* $\ell \equiv \mu + a$. *Then there exists* $C > 0$ *such that*

$$|E(n, \mu, a, \ell, \xi)| \leqslant \begin{cases} C \dfrac{|a|\, |\mu|^{r'-1} + |a|^{r'}}{n^{r'}} & \text{if} \quad \ell = \mu + a, \\ C \dfrac{|a|\, |\mu|^{r'-1} + |a|^{r'}}{|\ell|^{r'}} & \text{if} \quad \ell \neq \mu + a. \end{cases}$$

Proof. We first prove the result for $\ell = \mu + a$. In this case the definition reduces to

$$E(n, \mu, a, \mu + a, \xi) = F\left(\xi, \frac{\mu + a}{n}\right) - F\left(\xi, \frac{\mu}{n}\right)$$

where

$$F(\xi, y) := D_0^{-1}(y)\, \overline{Z_{r'}(\xi, y)}, \qquad 0 \leqslant |y| \leqslant \tfrac{3}{4}. \tag{A1}$$

Now Lemmata 4.1, 5.1, and 5.2, together with (4.8) and (4.9), tell us that

$$F(\xi, y) = 1 + y^{r'} G(\xi, y), \qquad 0 \leqslant |y| \leqslant \tfrac{3}{4},$$

where for $\xi \in \mathbb{R}$, $G(\xi, y)$ is a differentiable (indeed C^∞) function of y, with the derivative with respect to y bounded uniformly in ξ and y. By the mean-value theorem, there exists θ satisfying $0 < \theta < 1$, (with θ depending on ξ, μ, a, n) such that

$$F\left(\xi, \frac{\mu + a}{n}\right) - F\left(\xi, \frac{\mu}{n}\right) = \frac{a}{n} \frac{\partial F}{\partial y}\left(\xi, \frac{\mu + \theta a}{n}\right).$$

Since

$$\frac{\partial F}{\partial y}(\xi, y) = r' y^{r'-1} G(\xi, y) + y^{r'} \frac{\partial G}{\partial y}(\xi, y),$$

it follows that

$$\left| F\left(\xi, \frac{\mu + a}{n}\right) - F\left(\xi, \frac{\mu}{n}\right) \right| \leqslant C \left|\frac{a}{n}\right| \left|\frac{\mu + \theta a}{n}\right|^{r'-1}, \tag{A2}$$

with C independent of ξ, μ, a and n. With the aid of the standard inequality

$$(p + q)^m \leqslant 2^m (p^m + q^m), \qquad p, q, m > 0,$$

we obtain the desired result for the case $\ell = \mu + a$,

$$|E(n, \mu, a, \mu + a, \xi)| \leqslant C \frac{|a|\, |\mu|^{r'-1} + |a|^{r'}}{n^{r'}}.$$

For $\ell \neq \mu + a$ we may write

$$E(n, \mu, a, \ell, \xi) = E_1(n, \mu, a, \ell, \xi) + E_2(n, \mu, a, \ell, \xi),$$

where

$$E_1(n, \mu, a, \ell, \xi) := \left(\frac{n}{\ell}\right)^{r'} \left[H\left(\xi, \frac{\mu + a}{n}\right) - H\left(\xi, \frac{\mu}{n}\right)\right],$$

$$H(\xi, y) := y^{r'} F(\xi, y);$$

and

$$E_2(n, \mu, a, \ell, \xi) := -D_0^{-1}\left(\frac{\mu}{n}\right) \overline{Z_{r'}\left(\xi, \frac{\mu}{n}\right)}\, \mu^{r'} \left[\frac{1}{(\ell - a)^{r'}} - \frac{1}{\ell^{r'}}\right].$$

For the first term we use

$$\frac{\partial H(\xi, y)}{\partial y} = y^{r'-1}\left[y \frac{\partial F(\xi, y)}{\partial y} + r' F(\xi, y)\right],$$

so that from the mean-value theorem there exists θ satisfying $0 < \theta < 1$ such that

$$\begin{aligned} |E_1(n, \mu, a, \ell, \xi)| &= \left| \left(\frac{n}{\ell}\right)^{r'} \frac{a}{n} \frac{\partial}{\partial y} H\left(\xi, \frac{\mu + \theta a}{n}\right)\right| \\ &\leqslant C \left|\frac{n}{\ell}\right|^{r'} \left|\frac{a}{n}\right| \left|\frac{\mu + \theta a}{n}\right|^{r'-1} \\ &\leqslant C \frac{|a|}{|\ell|^{r'}} (|\mu|^{r'-1} + |\theta a|^{r'-1}) \\ &\leqslant C \frac{|a|\, |\mu|^{r'-1} + |a|^{r'}}{|\ell|^{r'}}. \end{aligned}$$

For the second term we have

$$|E_2(n, \mu, a, \ell, \xi)| \leqslant C\,|\mu|^{r'} \left|\frac{1}{(\ell - a)^{r'}} - \frac{1}{\ell^{r'}}\right|$$
$$= C\left|\frac{\mu}{\ell}\right|^{r'} \left|J\left(\frac{a}{\ell}\right) - J(0)\right|$$

where

$$J(y) := \frac{1}{(1-y)^{r'}}, \qquad y \leqslant \frac{1}{2}.$$

Note that under the conditions in the lemma, if $\ell \neq \mu + a$ then $a/\ell \leqslant \frac{1}{2}$. If $|\ell| \geqslant n/2$ this is straightforward, since $|a| \leqslant n/4$, implying $|a/\ell| \leqslant (n/4)(2/n) = \frac{1}{2}$. On the other hand, if $0 < |\ell| < n/2$ then it follows from $\ell \equiv \mu + a$ combined with $\ell \neq \mu + a$ that $\mu + a \notin \Lambda_h$. In turn it follows that μ and a must be of the same sign (since $|\mu + a| > |\mu|$), and that ℓ must be of opposite sign. (For example, if μ and a are both positive we must have $\mu + a > n/2$, and then $\ell = \mu + a - n < 0$ is necessary to ensure $|\ell| < n/2$.) Thus $a/\ell < 0 < \frac{1}{2}$. We also note that in this case $|\ell| = n - |\mu + a| \geqslant n - (n/2) - (n/4) = n/4$, so that in all cases

$$|\ell| \geqslant n/4 \qquad \text{when} \quad \ell \neq \mu + a.$$

From the mean-value theorem we now have, for some θ satisfying $0 < \theta < 1$,

$$|E_2(n, \mu, a, \ell, \xi)| = C\left|\frac{\mu}{\ell}\right|^{r'} \left|\frac{a}{\ell} J'\left(\theta \frac{a}{\ell}\right)\right|$$
$$= C\left|\frac{\mu}{\ell}\right|^{r'} \left|\frac{a}{\ell}\right| \frac{r'}{(1 - \theta(a/\ell))^{r'+1}},$$

and since the last factor is at most $r' 2^{r'+1}$,

$$|E_2(n, \mu, a, \ell, \xi)| \leqslant C\left|\frac{\mu}{\ell}\right|^{r'} \left|\frac{a}{\ell}\right| \leqslant C\frac{|\mu|^{r'-1}\,|a|}{|\ell|^{r'}},$$

where in the last step we used $|\mu/\ell| \leqslant (n/2)(4/n) = 2$ to remove one power of $|\mu/\ell|$. On combining the estimates for E_1 and E_2 the proof for the case $\ell \neq \mu + a$ is complete. Thus Lemma A2 holds. ∎

LEMMA A3. *Let*

$$K(n, a, \ell, \xi) := \delta_{\ell 0} - D_0^{-1}\left(-\frac{a}{n}\right) \overline{Z_{r'}\left(\xi, -\frac{a}{n}\right)} \left(-\frac{a}{\ell - a}\right)^{r'},$$

where $|a| \leqslant n/4$, *and* $\ell \equiv 0$. *There exists* $C > 0$ *such that*

$$|K(n, a, \ell, \xi)| \leqslant \begin{cases} C\left|\frac{a}{n}\right|^{r'} & \text{if } \ell = 0, \\ C\left|\frac{a}{\ell}\right|^{r'} & \text{if } \ell \neq 0. \end{cases}$$

Proof. If $\ell = 0$ the definition reduces to

$$K(n, a, 0, \xi) = F(\xi, 0) - F\left(\xi, -\frac{a}{n}\right),$$

where F is defined by (A1). The result then follows from (A2). For $\ell \neq 0$ it follows from (4.11) and Lemma 5.1 that

$$|K(n, a, \ell, \xi)| \leqslant C\left|\frac{a}{\ell - a}\right|^{r'} \leqslant C\left|\frac{a}{\ell}\right|^{r'},$$

where the last step follows from

$$|\ell - a| \geqslant |\ell| - |a| \geqslant 3\,|\ell|/4,$$

given $|\ell| \geqslant n$, $|a| \leqslant n/4$. ∎

LEMMA A4. *If* $\mu \in \Lambda_h$, $|a| \leqslant n/4$ *and* $\mu \not\equiv -a$, *and if* $\alpha > 1$, *then there exists* $C > 0$ *such that*

$$\sum_{\substack{\ell \equiv \mu + a, \\ \ell \neq \mu + a}} |\ell|^{-\alpha} \leqslant C n^{-\alpha}.$$

Proof. If $\mu + a \in \Lambda_h^*$ then the result follows from Lemma 4.3(a). If $\mu + a = 0$ the result is trivial. If $\mu + a \notin \Lambda_h$ then the sum may be rewritten as

$$\begin{aligned} \sum_{\substack{\ell \equiv \mu + a \\ \ell \neq \mu + a}} |\ell|^{-\alpha} &= \sum_{\substack{\ell \equiv \mu + a \\ \ell \neq (\mu + a)(n)}} |\ell|^{-\alpha} + |(\mu + a)(n)|^{-\alpha} - |\mu + a|^{-\alpha} \\ &\leqslant \sum_{\substack{\ell \equiv \mu + a \\ \ell \neq (\mu + a)(n)}} |\ell|^{-\alpha} + |(\mu + a)(n)|^{-\alpha}. \end{aligned}$$

The first term is bounded by $Cn^{-\alpha}$ by Lemma 4.3(a). For the second term note that under the circumstances of the lemma and for $\mu + a \notin \Lambda_h$ we have

$$(\mu + a)(n) = \mu + a \pm n,$$

with the $+$ sign if μ and a are both negative, and the $-$ sign if μ and a are both positive. Thus

$$|(\mu + a)(n)| = n - |\mu + a| \geqslant n - |\mu| - |a| \geqslant n - \frac{n}{2} - \frac{n}{4} = \frac{n}{4},$$

from which it follows that $|(\mu + a)(n)|^{-\alpha} \leqslant Cn^{-\alpha}$, and the result is proved. ▮

LEMMA A5. *For* $\alpha, t \in \mathbb{R}$,

$$\sum_{\mu \in \Lambda_h^*} |\hat{v}(\mu)|^2 \, |\mu|^{2\alpha} \leqslant n^{2 \max(\alpha - t, 0)} \, \|v\|_t^2.$$

Proof. Suppose first that $\alpha \leqslant t$. Then noting that $|\mu| \geqslant 1$ we use

$$\sum_{\mu \in \Lambda_h^*} |\hat{v}(\mu)|^2 \, |\mu|^{2\alpha} \leqslant \sum_{\mu \in \Lambda_h^*} |\hat{v}(\mu)|^2 \, |\mu|^{2t} \leqslant \|v\|_t^2.$$

Alternatively, suppose that $\alpha > t$. Then noting that $|\mu| < n$ we use

$$\sum_{\mu \in \Lambda_h^*} |\hat{v}(\mu)|^2 \, |\mu|^{2\alpha} = n^{2\alpha} \sum_{\mu \in \Lambda_h^*} |\hat{v}(\mu)|^2 \left|\frac{\mu}{n}\right|^{2\alpha}$$

$$\leqslant n^{2\alpha} \sum_{\mu \in \Lambda_h^*} |\hat{v}(\mu)|^2 \left|\frac{\mu}{n}\right|^{2t} \leqslant n^{2(\alpha - t)} \, \|v\|_t^2. \quad ▮$$

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